SOME MORE IDENTITIES OF THE ROGERS-RAMANUJAN TYPE

DOUGLAS BOWMAN AND JAMES MC LAUGHLIN

Abstract. In this we paper we prove several new identities of the Rogers-Ramanujan type. These identities were found as the result of a computer search. The proofs involve a variety of techniques, including constant-term methods, a theorem of Watson on basic hypergeometric series and the method of q-difference equations.

We briefly consider some related partition identities related to two of these identities.

1. introduction

The most famous of the "series = product" identities are the Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+1})(1-q^{5j+4})}, \tag{1.1}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+2})(1-q^{5j+3})},$$

These identities have a curious history ([15], p. 28). They were first proved by L.J. Rogers in 1894 ([20]) in a paper that was completely ignored. They were rediscovered (without proof) by Ramanujan sometime before 1913. In 1917, Ramanujan rediscovered Roger's paper. Also in 1917, these identities were rediscovered and proved independently by Issai Schur ([22]). There are now many different proofs.

There are numerous identities that are similar to the Rogers-Ramanujan identities. These include identities by Jackson [16], Rogers [20] and [21] and Bailey [7] and [8]. Of special note is Slater's 1952 paper [26], which contains a list of 130 such identities, many of them new (see the paper of Sills [23], for a corrected and annotated version of Slater's list). There are also other identities of Rogers-Ramanujan type in the literature.

Date: July 7 2006.

2000 *Mathematics Subject Classification.* Primary: 33D15. Secondary:05A17, 05A19, 11B65, 11P81, 33F10.

Key words and phrases. q-series, Rogers-Ramanujan identities, Slater's Identities.

The research of the first author was partially supported by National Science Foundation grant DMS-0300126.

In this present paper we prove several new identities, found through a computer search. We use various methods to prove these identities, including constant-term results (similar those used by Andrews in [2]), a theorem of Watson on basic hypergeometric series which transforms an ${}_8\phi_7$ into a ${}_4\phi_3$, and the method of q-difference equations.

Two identities on Slater's list, numbers 34 and 36, can be stated as follows:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}.$$

These identities became better known when they were given partition interpretations by Göllnitz [13] and, independently, by Gordon [14]. Our investigations turned up a companion identity. Our identity is the following:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^2;q^8)_\infty (q^3;q^8)_\infty (q^7;q^8)_\infty}.$$

We first thought this identity was new as we had not seen it explicitly in print, but Andrew Sills pointed out that it follows as a special case of Corollary 2.7 on page 21 of [1]. Sills also drew our attention to another companion identity,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-1/q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty (q^5;q^8)_\infty (q^6;q^8)_\infty},$$

This identity also follows as a special case of Corollary 2.7 on page 21 of [1].

As with the Göllnitz-Gordon identities, the analytic identities above also have interpretations in terms of partitions.

Theorem 4 *The number of partitions of any positive integer n into parts $\equiv 2, 3$ or $7 \pmod 8$ is equal to the number of partitions of the form*

$$n = n_1 + n_2 + \dots n_k,$$

where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$, if n_i is odd $(1 \leq i \leq k-1)$, and $n_k \geq 2$.

Theorem 5 *The number of partitions of any positive integer n into parts $\equiv 1, 5$ or $6 \pmod 8$ is equal to the number of partitions of the form*

$$n = n_1 + n_2 + \dots n_k,$$

where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$, if n_i is odd $(1 \leq i \leq k-1)$.

Since these partition results do not appear to be well known, we include them with the aim of bringing them to the attention of a wider audience.

2. Constant term results

We begin by recalling the following results [1], pp. 35–36.

Lemma 1. *If* $\begin{bmatrix} n \\ m \end{bmatrix}$ *denotes the Gaussian polynomial defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, & \text{if } 0 \le m \le n, \\ 0, & \text{otherwise} , \end{cases}$$

then

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \tag{2.1}$$

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j.$$

Upon letting $N \to \infty$ we get

$$(z;q)_\infty = \sum_{j=0}^{\infty} \frac{(-1)^j z^j q^{j(j-1)/2}}{(q;q)_j}, \tag{2.2}$$

$$\frac{1}{(z;q)_\infty} = \sum_{j=0}^{\infty} \frac{z^j}{(q;q)_j}.$$

The above identities are all special cases of the q-analog of the binomial series due to Cauchy [11]:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z|<1, \ |q|<1. \tag{2.3}$$

We also use the following identity, which is a special case of Ramanujan's ${}_1\psi_1$ summation formula, and can be found in [2]:

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2} t^n}{(b;q)_n} = \frac{(t;q)_\infty (q/t;q)_\infty (q;q)_\infty}{(b/t;q)_\infty (b;q)_\infty}. \tag{2.4}$$

Note that the special case $b=0$ gives the Jacobi triple product identity:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} t^n = (t;q)_\infty (q/t;q)_\infty (q;q)_\infty. \tag{2.5}$$

We now use these results to prove some transformations which will later be used to derive some new identities of the Rogers-Ramanujan type. Theorem 1 was motivated in part by Andrews proofs in [2] of eight identities due to Rogers ([20] and [21]).

Theorem 1. *Let a, b, γ and $q \in \mathbb{C}$, $|q| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(b;q)_n(q;q)_n} = \frac{(-\gamma;q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(-a\gamma/b;q)_n q^{n(n-1)/2}(-b)^n}{(-\gamma;q)_n(q;q)_n}, \tag{2.6}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(q;q)_n} = (-\gamma;q)_\infty \sum_{n=0}^{\infty} \frac{(-a\gamma)^n q^{n(n-1)}}{(-\gamma;q)_n(q;q)_n}, \tag{2.7}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(-a\gamma;q)_n(q;q)_n} = \frac{(-\gamma;q)_\infty}{(-a\gamma;q)_\infty}. \tag{2.8}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n(n-1)/2}\gamma^n}{(\gamma;q^2)_n(q;q)_n} = \frac{1}{(\gamma;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2-n}\gamma^n}{(q^2;q^2)_n}. \tag{2.9}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-\gamma)^n}{(q;q)_n}. \tag{2.10}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}(-\gamma)^n}{(\gamma/q;q^2)_n(q^2;q^2)_n} = \frac{1}{(\gamma/q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-2n}(-\gamma)^n}{(q;q)_n}. \tag{2.11}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\gamma^n}{(\gamma;q)_n(q;q)_n} = \frac{1}{(\gamma;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2-n}\gamma^{2n}}{(q^2;q^2)_n}. \tag{2.12}$$

Remark: The identity at (2.6) is due to Ramanujan in the form

$$(-aq)_\infty \sum_{j=0}^{\infty} \frac{(bq)^j(-c/b)_j\, q^{j(j-1)/2}}{(q)_j(-aq)_j} = (-bq)_\infty \sum_{j=0}^{\infty} \frac{(aq)^j(-c/a)_j\, q^{j(j-1)/2}}{(q)_j(-bq)_j}. \tag{2.13}$$

This identity is found in Ramanujan's lost notebook [18] and a proof can be found in the recent book by Andrews and Berndt [5]. However, we wish to give an independent proof.

Proof. The transformations below from infinite series to infinite products and back are the result of using (2.3) and (2.4), unless stated otherwise.

We now prove (2.6). We assume below that $|\gamma| < |z|$.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n-1)/2}\gamma^n}{(b;q)_n(q;q)_n} \\
&= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty} \frac{(-1)^n(-z)^n q^{n(n-1)/2}}{(b;q)_n} \sum_{m=0}^{\infty} \frac{(a;q)_m(\gamma/z)^m}{(q;q)_m} \\
&= \text{coeff. of } z^0 \text{ in } \frac{(-z;q)_\infty(-q/z;q)_\infty(q;q)_\infty}{(-b/z;q)_\infty(b;q)_\infty} \frac{(a\gamma/z;q)_\infty}{(\gamma/z;q)_\infty}
\end{aligned}$$

$$= \text{coeff. of } z^0 \text{ in } \frac{(-z;q)_\infty(-q/z;q)_\infty(q;q)_\infty}{(\gamma/z;q)_\infty(-\gamma;q)_\infty}\frac{(a\gamma/z;q)_\infty}{(-b/z;q)_\infty}\frac{(-\gamma;q)_\infty}{(b;q)_\infty}$$
$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty}\frac{(-1)^n(-z)^n q^{n(n-1)/2}}{(-\gamma;q)_n}\sum_{m=0}^{\infty}\frac{(-a\gamma/b;q)_m(-b/z)^m}{(q;q)_m} \times \frac{(-\gamma;q)_\infty}{(b;q)_\infty}$$
$$= \frac{(-\gamma;q)_\infty}{(b;q)_\infty}\sum_{n=0}^{\infty}\frac{(-a\gamma/b;q)_n q^{n(n-1)/2}(-b)^n}{(-\gamma;q)_n(q;q)_n}.$$

The identity at (2.7) follows from (2.6), upon letting $b \to 0$ and (2.8) upon letting $b = -a\gamma$ in (2.6).

We next prove (2.9). Note that (2.2), rather than (2.3), is used for some series-product transformations in this case, as is the case in the proof of (2.10).

$$\sum_{n=0}^{\infty}\frac{q^{3n(n-1)/2}\gamma^n}{(\gamma;q^2)_n(q;q)_n}$$
$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty}\frac{(-1)^n(-z)^n q^{n(n-1)}}{(\gamma;q^2)_n}\sum_{m=0}^{\infty}\frac{q^{m(m-1)/2}(\gamma/z)^m}{(q;q)_m}$$
$$= \text{coeff. of } z^0 \text{ in } \frac{(-z;q^2)_\infty(-q^2/z;q^2)_\infty(q^2;q^2)_\infty}{(-\gamma/z;q^2)_\infty(\gamma;q^2)_\infty}(-\gamma/z;q)_\infty$$
$$= \text{coeff. of } z^0 \text{ in } \frac{(-z;q^2)_\infty(-q^2/z;q^2)_\infty(q^2;q^2)_\infty}{(\gamma;q^2)_\infty}(-\gamma q/z;q^2)_\infty$$
$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty}\frac{(-1)^n(-z)^n q^{n(n-1)}}{(\gamma;q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{m(m-1)}(\gamma q/z)^m}{(q^2;q^2)_m}$$
$$= \frac{1}{(\gamma;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{2n^2-n}\gamma^n}{(q^2;q^2)_n}.$$

For (2.10),

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(-\gamma)^n}{(\gamma q;q^2)_n(q^2;q^2)_n}$$
$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty}\frac{(-1)^n z^n q^{n(n-1)}}{(\gamma q;q^2)_n}\sum_{m=0}^{\infty}\frac{(\gamma/z)^m}{(q^2;q^2)_m}$$
$$= \text{coeff. of } z^0 \text{ in } \frac{(z;q^2)_\infty(q^2/z;q^2)_\infty(q^2;q^2)_\infty}{(\gamma q/z;q^2)_\infty(\gamma q;q^2)_\infty}\frac{1}{(\gamma/z;q^2)_\infty}$$
$$= \text{coeff. of } z^0 \text{ in } \frac{(z;q^2)_\infty(q^2/z;q^2)_\infty(q^2;q^2)_\infty}{(\gamma q;q^2)_\infty}\frac{1}{(\gamma/z;q)_\infty}$$

$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty} \frac{(-1)^n z^n q^{n(n-1)}}{(\gamma q; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(\gamma/z)^m}{(q;q)_m}$$

$$= \frac{1}{(\gamma q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-\gamma)^n}{(q;q)_n}.$$

For (2.11),

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)}(-\gamma)^n}{(\gamma/q; q^2)_n (q^2;q^2)_n}$$

$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty} \frac{(-1)^n z^n q^{n(n-1)}}{(\gamma/q; q^2)_n} \sum_{m=0}^{\infty} \frac{(\gamma/z)^m}{(q^2;q^2)_m}$$

$$= \text{coeff. of } z^0 \text{ in } \frac{(z;q^2)_\infty (q^2/z;q^2)_\infty (q^2;q^2)_\infty}{(\gamma/qz;q^2)_\infty (\gamma/q;q^2)_\infty} \frac{1}{(\gamma/z;q^2)_\infty}$$

$$= \text{coeff. of } z^0 \text{ in } \frac{(z;q^2)_\infty (q^2/z;q^2)_\infty (q^2;q^2)_\infty}{(\gamma/q;q^2)_\infty} \frac{1}{(\gamma/zq;q)_\infty}$$

$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty} \frac{(-1)^n z^n q^{n(n-1)}}{(\gamma/q; q^2)_\infty} \sum_{m=0}^{\infty} \frac{(\gamma/zq)^m}{(q;q)_m}$$

$$= \frac{1}{(\gamma/q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2-2n}(-\gamma)^n}{(q;q)_n}.$$

We next prove (2.12)

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}\gamma^n}{(\gamma;q)_n (q;q)_n}$$

$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-z)^n q^{n(n-1)/2}}{(\gamma;q)_n} \sum_{m=0}^{\infty} \frac{(\gamma/z)^m}{(q;q)_m}$$

$$= \text{coeff. of } z^0 \text{ in } \frac{(-z;q)_\infty (-q/z;q)_\infty (q;q)_\infty}{(-\gamma/z;q)_\infty (\gamma;q)_\infty} \frac{1}{(\gamma/z;q)_\infty}$$

$$= \text{coeff. of } z^0 \text{ in } \frac{(-z;q)_\infty (-q/z;q)_\infty (q;q)_\infty}{(\gamma;q)_\infty} \frac{1}{((\gamma/z)^2;q^2)_\infty}$$

$$= \text{coeff. of } z^0 \text{ in } \sum_{n=-\infty}^{\infty} \frac{(-1)^n (-z)^n q^{n(n-1)/2}}{(\gamma;q)_\infty} \sum_{m=0}^{\infty} \frac{(\gamma/z)^{2m}}{(q^2;q^2)_m}$$

$$= \frac{1}{(\gamma;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2-n}\gamma^{2n}}{(q^2;q^2)_n}.$$

□

As regards identities of the Rogers-Ramanujan type, what each of these transformations imply is that if one side has a product representation for particular values of the parameters, then so does the other side. Indeed

several identities on Slater's list can be derived from other identities in this way. We do not pursue that here, instead using Theorem 1 to prove some identities of the Rogers-Ramanujan type, which we believe to be new. In all cases we assume $|q| < 1$.

Corollary 1.
$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(q^6;q^6)_\infty}{(-q^3;q^6)_\infty (q^4;q^4)_\infty}.$$

Proof. We use the following identity (Identity (28) from Slater's list):

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2+n}}{(q;q)_{2n+1}} = \frac{(-q;q^6)_\infty (-q^5;q^6)_\infty (q^6;q^6)_\infty (-q^2;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.14}$$

In (2.6), replace q by q^2 and let $a = q$, $b = -q^2$ and $\gamma = -q^3$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{n^2-n}(-q^3)^n}{(-q^2;q^2)_n (q^2;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2-n}(q^2)^n}{(q^3;q^2)_n (q^2;q^2)_n} \frac{(q^3;q^2)_\infty}{(-q^2;q^2)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2+n}}{(q;q)_{2n+1}} \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty} \\
&= \frac{(-q;q^6)_\infty (-q^5;q^6)_\infty (q^6;q^6)_\infty (-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty} \quad \text{(by (2.14))} \\
&= \frac{(q^6;q^6)_\infty}{(-q^3;q^6)_\infty (q^4;q^4)_\infty}.
\end{aligned}$$

□

Corollary 2.
$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.$$

Proof. In (2.8), replace q by q^2 and set $a = -q^2$ and $\gamma = q$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} &= \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q^3;q^2)_n (q^2;q^2)_n} \\
&= \frac{1}{1-q} \frac{(-q;q^2)_\infty}{(q^3;q^2)_\infty} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}.
\end{aligned}$$

□

Corollary 3.
$$\sum_{n=0}^{\infty} \frac{(-1;q)_{2n} q^{n^2+n}}{(q^2;q^2)_n (q^2;q^4)_n} = \frac{(-q^3;q^6)_\infty^2 (q^6;q^6)_\infty (-q^2;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Proof. We use the following identity ((25) from Slater's list, with q replaced by $-q$):

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(q^4;q^4)_n}=\frac{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.15}$$

In (2.6), replace q by q^2 and set $a=-1$, $b=q$ and $\gamma=q^2$. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1;q)_{2n}q^{n^2+n}}{(q^2;q^2)_n(q^2;q^4)_n}\\
&=\sum_{n=0}^{\infty}\frac{(-1;q^2)_n(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n(q;q^2)_n(-q;q^2)_n}\\
&=\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2-n}(q^2)^n}{(q;q^2)_n(q^2;q^2)_n}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n(q^2;q^2)_n}\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(q^4;q^4)_n}\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\\
&=\frac{(-q^3;q^6)_\infty^2(q^6;q^6)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\quad\text{(by (2.15))}.
\end{aligned}$$

The result now follows. □

Corollary 4.

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q^2;q^8)_\infty(q^3;q^8)_\infty(q^7;q^8)_\infty}. \tag{2.16}$$

Proof. In (2.7) replace q by q^2 and set $a=-q$ and $\gamma=q^2$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-q;q^2)_n}{(q^2;q^2)_n}&=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2-n}(q^2)^n}{(q^2;q^2)_n}\\
&=(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{(q^3)^n q^{2n^2-2n}}{(-q^2;q^2)_n(q^2;q^2)_n}\\
&=(-q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{(q^3)^n(q^4)^{n(n-1)/2}}{(q^4;q^4)_n}\\
&=(-q^2;q^2)_\infty(-q^3;q^4)_\infty\\
&=\frac{1}{(q^2;q^8)_\infty(q^3;q^8)_\infty(q^7;q^8)_\infty}.
\end{aligned}$$

□

We had initially thought this identity was new, since we had not seen it explicitly in print, but it follows easily as a special case of Corollary 2.7 on page 21 of [1]. We include it here because apparently the partition interpretations of this identity and another identity (see below) are not widely known.

This identity is interesting for a number of reasons. Firstly, it is clearly a companion identity to the Göllnitz-Gordon identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}, \tag{2.17}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}. \tag{2.18}$$

Secondly, in identities of the Rogers-Ramanujan type, it is usual for the powers of q in the product side to occur in pairs that sum to the modulus or to be exactly half the modulus. The Göllnitz-Gordon identities illustrate this nicely: the modulus is 8 and the powers of q are $(1,4,7)$ and $(3,4,5)$. However the identity at (2.16) breaks this pattern with its powers being $(2,3,7)$ (although still summing to 12).

Partition identities corresponding to (2.17) and (2.18) were given by Göllnitz [13] and Gordon [14], independently.

Theorem 2. (*Theorem 2 in* [14]) *The number of partitions of any positive integer n into parts $\equiv 1, 4$ or $7(\mod 8)$ is equal to the number of partitions of the form*

$$n = n_1 + n_2 + \dots n_k,$$

where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$, if n_i is even $(1 \leq i \leq k-1)$.

Theorem 3. (*Theorem 3 in* [14]) *The number of partitions of any positive integer n into parts $\equiv 3, 4$ or $5(\mod 8)$ is equal to the number of partitions of the form*

$$n = n_1 + n_2 + \dots n_k,$$

satisfying $n_k \geq 3$, in addition to the conditions of Theorem 2.

We next prove a partition identity deriving from (2.16). This partition result is apparently not well known, as it, and another result below, are rarely mentioned, despite being on the same level as the Göllnitz-Gordon identities.

Theorem 4. *The number of partitions of any positive integer n into parts $\equiv 2, 3$ or $7(\mod 8)$ is equal to the number of partitions of the form*

$$n = n_1 + n_2 + \dots n_k,$$

where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$, if n_i is odd $(1 \leq i \leq k-1)$, and $n_k \geq 2$.

Proof. The proof follows similar lines to that given by Gordon in [14] for the partition identities corresponding to (2.17) and (2.18).

In this paper, Gordon considered the continued fraction

$$F(a,q) := 1 + aq + \frac{aq^2}{1+aq^3} + \frac{aq^4}{1+aq^5} + \frac{aq^6}{1+aq^7} + \cdots$$

and showed that the numerators converged to

$$P(a,q) := \sum_{n=0}^{\infty} \frac{q^{n^2} a^n (-q;q^2)_n}{(q^2;q^2)_n}.$$

We have replaced Gordon's x with q to be consistent with our present notation. If we set $a = q$ above, then we get that $P(q,q)$ equals the left side of (2.16). Let $P_v(q) = P_v$ denote the v-th numerator convergent of

$$1 + q^2 + \frac{q^3}{1+q^4} + \frac{q^5}{1+q^6} + \frac{q^7}{1+q^8} + \cdots.$$

It is clear that $P_0 = 1 + q^2$, $P_1 = 1 + q^2 + q^3 + q^4 + q^6$ and

$$P_v = (1 + q^{2v+2})P_{v-1} + q^{2v+1}P_{2v-2}, \qquad v \geq 2. \tag{2.19}$$

Let $c_v(n)$ denote the number of partitions of n of the form

$$n = n_1 + n_2 + \ldots n_k,$$

where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$, if n_i is odd $(1 \leq i \leq k-1)$, $n_k \geq 2$ and $n_1 \leq 2v + 2$. For ease of notation, we call these *partitions of type* v. Set

$$S_v(q) = \sum_{n=0}^{\infty} c_v(n) q^n.$$

If it can be shown that $S_v(q) = P_v(q)$ for all non-negative integers v, then it will follow that $P(q,q)$ is the generating function for the number of partitions of n of the form

$$n = n_1 + n_2 + \ldots n_k,$$

where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$, if n_i is odd $(1 \leq i \leq k-1)$, and $n_k \geq 2$. The theorem will then be true by (2.16).

It is easy to check that $S_v(q) = P_v(q)$ holds for $v = 0, 1$, so that all that is necessary is to show that $S_v(q)$ satisfies the same recurrence relation as $P_v(q)$, namely (2.19). Upon substituting the corresponding series into (2.19), it becomes apparent that this is equivalent to showing that

$$c_v(n) = c_{v-1}(n) + c_{v-1}(n - 2v - 2) + c_{v-2}(n - 2v - 1).$$

It is not difficult to see that $c_{v-1}(n)$ equals the number of partitions of type v with $n_1 \leq 2v$, that $c_{v-1}(n - 2v - 2)$ equals the number of partitions of type v with $n_1 = 2v + 2$ and that $c_{v-2}(n - 2v - 1)$ equals the number of partitions of type v with $n_1 = 2v + 1$. The result now follows. $\square$

Andrew Sills drew our attention to another “companion” identity, namely

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-1/q;q^2)_n}{(q^2;q^2)_n} = \frac{1}{(q;q^8)_\infty(q^5;q^8)_\infty(q^6;q^8)_\infty}. \tag{2.20}$$

Sills pointed out that this analytic identity also has an interpretation in terms of partitions, which also does not appear to be widely known. We state it here for the purpose of making it more widely known.

Theorem 5. *The number of partitions of any positive integer n into parts $\equiv 1, 5$ or $6 (\mod 8)$ is equal to the number of partitions of the form*

$$n = n_1 + n_2 + \dots n_k,$$

where $n_i \geq n_{i+1} + 2$, and $n_i \geq n_{i+1} + 3$, if n_i is odd $(1 \leq i \leq k-1)$.

Corollary 5.

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+n}}{(q;q)_{2n+1}} = \frac{1}{(q;q^4)_\infty(q^2;q^4)_\infty(q^3;q^4)_\infty}. \tag{2.21}$$

Proof. In (2.8) let $a = -q$ and $\gamma = q^2$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+n}}{(q;q)_{2n+1}} &= \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2-n}(q^2)^n}{(q^3;q^2)_n(q^2;q^2)_n} \\
&= \frac{1}{1-q} \frac{(-q^2;q^2)_\infty}{(q^3;q^2)_\infty} \\
&= \frac{1}{(q;q^4)_\infty(q^2;q^4)_\infty(q^3;q^4)_\infty}.
\end{aligned}$$

□

Corollary 6.

$$\sum_{n=0}^{\infty} \frac{q^{(3n^2+3n)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{1}{(q;q^2)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}. \tag{2.22}$$

Proof. In (2.9) let $\gamma = q^3$. Then

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{(3n^2+3n)/2}}{(q;q^2)_{n+1}(q;q)_n} &= \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{(3n^2-3n)/2}(q^3)^n}{(q^3;q^2)_n(q;q)_n} \\
&= \frac{1}{1-q} \frac{1}{(q^3;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2-n}(q^3)^n}{(q^2;q^2)_n} \\
&= \frac{1}{(q;q^2)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}.
\end{aligned}$$

□

The last equality follows from the second Rogers-Ramanujan identity (1.1).

3. Identities of the Rogers-Ramanujan type via Watson's Theorem

An ${}_r\phi_s$ basic hypergeometric series is defined by

$$
{}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n (b_1;q)_n \ldots (b_s;q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{s+1-r} x^n,
$$

for $|q| < 1$.

Watson's theorem is the following.

Theorem 6.

$$
\tag{3.1} {}_8\phi_7\left(\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-n} \\ \sqrt{a},\ -\sqrt{a},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{n+1} \end{matrix}; q, \frac{a^2 q^{n+2}}{bcde}\right) = \frac{(aq)_n (aq/de)_n}{(aq/d)_n (aq/e)_n}\ {}_4\phi_3\left(\begin{matrix} aq/bc, d, e, q^{-n} \\ aq/b, aq/c, deq^{-n}/a \end{matrix}; q, q\right),
$$

where n is a non-negative integer.

We will use some parts of the following obvious corollary later.

Corollary 7.

$$
\tag{3.2} \sum_{r\geq 0} \frac{(1-aq^{2r})(a)_r (b)_r (c)_r (d)_r (e)_r \left(-a^2/bcde\right)^r q^{r(r-1)/2+2r}}{(1-a)(aq/b)_r (aq/c)_r (aq/d)_r (aq/e)_r (q)_r} = \frac{(aq)_\infty (aq/de)_\infty}{(aq/d)_\infty (aq/e)_\infty} \sum_{r\geq 0} \frac{(aq/bc)_r (d)_r (e)_r (aq/de)^r}{(aq/b)_r (aq/c)_r (q)_r}. \ (n \to \infty \ \textit{in}\ (3.1))
$$

$$
\tag{3.3} \sum_{r\geq 0} \frac{(1-aq^{2r})(a)_r (c)_r (d)_r (e)_r \left(a^2/cde\right)^r q^{r(r-1)+2r}}{(1-a)(aq/c)_r (aq/d)_r (aq/e)_r (q)_r} = \frac{(aq)_\infty (aq/de)_\infty}{(aq/d)_\infty (aq/e)_\infty} \sum_{r\geq 0} \frac{(d)_r (e)_r (aq/de)^r}{(aq/c)_r (q)_r}. \ (n, b \to \infty \ \textit{in}\ (3.1))
$$

$$
\tag{3.4} \sum_{r\geq 0} \frac{(1-aq^{2r})(a)_r (c)_r (e)_r \left(-a^2/ce\right)^r q^{3r(r-1)/2+2r}}{(1-a)(aq/c)_r (aq/e)_r (q)_r} = \frac{(aq)_\infty}{(aq/e)_\infty} \sum_{r\geq 0} \frac{(e)_r (-aq/e)^r q^{r(r-1)/2}}{(q)_r (aq/c)_r}. \ (n, b, d \to \infty \ \textit{in}\ (3.1))
$$

Watson [27] used his transformation in his proof of the Rogers-Ramanujan identities (1.1).

Several other identities on Slater's list can be derived from other identities on the list, using special cases of Watson's transformation, but we do not

pursue that here either, instead once again using these transformations to prove identities which we believe to be new.

Corollary 8. *Let $c \in \mathbb{C}$. Then*

$$\sum_{r=0}^{\infty} \frac{(1-cq^r)(c^2;q)_r(-c)^r q^{r^2}}{(q;q)_r} = (c^2q;q^2)_\infty (c;q)_\infty. \tag{3.5}$$

Proof. We make use of the following result of Lebesgue (see [1], page 21):

$$\sum_{n=0}^{\infty} \frac{(a;q)_n q^{n(n+1)/2}}{(q;q)_n} = (aq;q^2)_\infty (-q;q)_\infty. \tag{3.6}$$

For ease of notation we write (3.4) as

$$\sum_{r\geq 0} \frac{(1-Aq^{2r})(A)_r(C)_r(E)_r \left(-A^2/CE\right)^r q^{3r(r-1)/2+2r}}{(1-A)(Aq/C)_r(Aq/E)_r(q)_r} = \frac{(Aq)_\infty}{(Aq/E)_\infty} \sum_{r\geq 0} \frac{(E)_r(-Aq/E)^r q^{r(r-1)/2}}{(q)_r(Aq/C)_r}.$$

If we replace C by $-c/a$ and set $A = c$ and $E = -c/b$ in (3.4), we get that

$$\sum_{r\geq 0} \frac{(1-cq^{2r})(-c/a)_r(-c/b)_r(c)_r\,(-ab)^r\, q^{3r(r-1)/2+2r}}{(1-c)(-aq)_r(-bq)_r(q)_r} = \frac{(cq)_\infty}{(-bq)_\infty} \sum_{j=0}^{\infty} \frac{(bq)^j(-c/b)_j\, q^{j(j-1)/2}}{(q)_j(-aq)_j}. \tag{3.7}$$

Now set $b = 1$ and let $a \to 0$ to get that

$$\begin{aligned}
\sum_{r\geq 0} \frac{(1-cq^{2r})(-c)_r(c)_r\,(-c)^r\, q^{2r^2}}{(1-c)(-q)_r(q)_r} &= \frac{(cq)_\infty}{(-q)_\infty} \sum_{j=0}^{\infty} \frac{(q)^j(-c)_j\, q^{j(j-1)/2}}{(q)_j} \\
\Rightarrow \sum_{r\geq 0} \frac{(1-cq^{2r})(c^2;q^2)_r\,(-c)^r\, q^{2r^2}}{(q^2;q^2)_r} &= \frac{(c)_\infty}{(-q)_\infty} \sum_{j=0}^{\infty} \frac{(-c)_j\, q^{j(j+1)/2}}{(q)_j} \\
&= \frac{(c)_\infty}{(-q)_\infty}(-cq;q^2)_\infty(-q;q)_\infty (\text{by } (3.6)) \\
&= (c^2q^2;q^4)_\infty(c;q^2)_\infty.
\end{aligned}$$

The result is now immediate upon replacing q^2 by q. $\square$

Corollary 9. *Let a, b and $q \in \mathbb{C}$ satisfy $|q| < \min\{|b|, 1\}$. Then*

$$\sum_{r=0}^{\infty} \frac{(1+aq^r)(a^2;q)_r(b;q)_r\,(-a/b)^r\,q^{r(r+1)/2}}{(a^2q/b;q)_r(q;q)_r} = \frac{(-a;q)_\infty(a^2q;q^2)_\infty(aq/b;q)_\infty}{(a^2q/b;q)_\infty}. \tag{3.8}$$

Proof. We will use Bailey's identity [6]:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(aq/b;q)_n(q;q)_n}\left(-\frac{q}{b}\right)^n = \frac{(aq;q^2)_\infty(-q;q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b;q)_\infty(-q/b;q)_\infty}. \tag{3.9}$$

In (3.3) replace a by $-a$, set $c=-b$, $d=a$ and $e=b$, so that (3.3) becomes

$$\sum_{r\geq 0} \frac{(1+aq^{2r})(-a)_r(-b)_r(a)_r(b)_r\left(-a/b^2\right)^r q^{r(r-1)+2r}}{(1+a)(aq/b)_r(-q)_r(-aq/b)_r(q)_r} = \frac{(-aq)_\infty(-q/b)_\infty}{(-q)_\infty(-aq/b)_\infty}\sum_{r\geq 0}\frac{(a)_r(b)_r(-q/b)^r}{(aq/b)_r(q)_r},$$

and the result now follows from (3.9), after replacing b^2 by b and q^2 by q. □

We next derive another transformation from (3.2), which we will use below. Replace b by $-a/b$ and d by $-a/d$, so that

$$\sum_{r\geq 0} \frac{(1-aq^{2r})(a)_r(-a/b)_r(c)_r(-a/d)_r(e)_r\,(-bd/ce)^r\,q^{r(r-1)/2+2r}}{(1-a)(-bq)_r(aq/c)_r(-dq)_r(aq/e)_r(q)_r} = \frac{(aq)_\infty(-dq/e)_\infty}{(-dq)_\infty(aq/e)_\infty}\sum_{r\geq 0}\frac{(-bq/c)_r(-a/d)_r(e)_r(-dq/e)^r}{(-bq)_r(aq/c)_r(q)_r}. \tag{3.10}$$

Now let $a \to 0$ to get

$$\sum_{r\geq 0} \frac{(c)_r(e)_r\,(-bd/ce)^r\,q^{r(r-1)/2+2r}}{(-bq)_r(-dq)_r(q)_r} = \frac{(-dq/e)_\infty}{(-dq)_\infty}\sum_{r\geq 0}\frac{(-bq/c)_r(e)_r(-dq/e)^r}{(-bq)_r(q)_r}. \tag{3.11}$$

Next let $e \to \infty$ and we have that

$$\sum_{r\geq 0} \frac{(c)_r\,(bd/c)^r\,q^{r(r-1)+2r}}{(-bq)_r(-dq)_r(q)_r} = \frac{1}{(-dq)_\infty}\sum_{r\geq 0}\frac{(-bq/c)_r(dq)^r q^{r(r-1)/2}}{(-bq)_r(q)_r}. \tag{3.12}$$

Corollary 10.

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty}{(q^2;q^2)_\infty}. \tag{3.13}$$

Remark: This is a companion identity to number (27) on Slater's list, with q replaced by $-q$:

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty}.$$

Proof. We first apply (3.12), with q replaced by q^2, $c = q$, $b = 1/q$ and $d = 1$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{2n^2}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} &= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(1/q^2)^n(q;q^2)_n}{(-q;q^2)_n(-q^2;q^2)_n(q^2;q^2)_n} \\
&= \frac{1}{(-q^2;q^2)_\infty} \sum_{r\geq 0} \frac{(-1;q^2)_r q^{r(r+1)}}{(-q;q^2)_r(q^2;q^2)_r} \\
&= \frac{1}{(-q;q^2)_\infty} \sum_{r\geq 0} \frac{(-q;q^2)_r q^{r(r+1)}}{(-q^2;q^2)_r(q^2;q^2)_r} \\
&= \frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty}{(q^2;q^2)_\infty}.
\end{aligned}$$

For the third equality we used (2.6), with q replaced by q^2, $a = -1$, $b = -q$ and $\gamma = q^2$. For the last equality, we used Identity (25) on Slater's list:

$$\sum_{r\geq 0} \frac{(-q;q^2)_r q^{r(r+1)}}{(-q^2;q^2)_r(q^2;q^2)_r} = \frac{(q^3;q^6)_\infty^2(q^6;q^6)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

$\square$

4. The Method of q-Difference Equations

The transformations used in previous sections do not appear to be applicable to the following identity, nor the one in Theorem 8 in the next section.

Theorem 7. *Let $|q| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{4.1}$$

Remark: This identity is clearly a companion to Identity (21) on Slater's list:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3;q^5)_\infty(-q^2;q^5)_\infty(q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{4.2}$$

To prove (4.1), we use the method of q-difference equations described by Andrews in [2] and used extensively by Sills in [23].

The proof of a "Series Σ = Product Π" identity by this method involves finding two representations for a sequence of polynomials in q, P_n (the "fermionic" representation) and Q_n (the "bosonic" representation) such that

$$\lim_{n\to\infty} P_n = \Sigma \qquad \text{and} \qquad \lim_{n\to\infty} Q_n = \Pi.$$

In addition, it needs to be shown (since it is almost always not obvious) that $P_n = Q_n$ for all non-negative integers n, this being attained by showing that P_n and Q_n satisfy the same q-difference equation and are equal for as many initial values as are required by the q-difference equation. These facts taken together imply that $\Sigma = \Pi$. Note that determining a possible form for Q_n usually involves intelligent guesswork and numerical computations to check the conjectured form - there is no algorithm that will automatically generate Q_n for a particular identity. We think the derivation of Q_n is also of interest in this case, so we include some details of this, as well as showing that the derived Q_n is actually the correct one.

We now describe certain polynomials which will be used to construct the Q_n for the identity in Theorem 7 (see the paper by Sill's [23], pp.6–10, for more on these and similar polynomials). Recall the q-binomial coefficients defined in Lemma 1.

$$\begin{aligned} T_0(L,A;q) &:= \sum_{r=0}^{L} (-1)^r \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_{q}, \\ T_1(L,A;q) &:= \sum_{r=0}^{L} (-q)^r \begin{bmatrix} L \\ r \end{bmatrix}_{q^2} \begin{bmatrix} 2L-2r \\ L-A-r \end{bmatrix}_{q}, \\ U(L,A;q) &:= T_0(L,A;q) + T_0(L,A+1,q). \end{aligned} \tag{4.3}$$

The following recurrences can be found in [4]. For $L \geq 1$,

$$\begin{aligned} T_0(L,A;q) &:= T_0(L-1,A-1;q) + q^{L+A} T_1(L-1,A;q), \\ &\qquad + q^{2L+2A} T_0(L-1,A+1;q), \\ T_1(L,A;q) &- q^{L-A} T_0(L,A;q) - T_1(L,A+1;q) \\ &\qquad + q^{L+A+1} T_0(L,A+1;q) = 0. \end{aligned} \tag{4.4}$$

The following relation can be found in the paper of Andrews [3]. For $L \geq 1$,

$$\begin{aligned} U(L,A;q) &= (1+q+q^{2L-1}) U(L-1,A;q) - qU(L-2,A;q) \\ &\quad + q^{2L-2A} T_0(L-2,A-2;q) + q^{2L+2A+2} T_0(L-2,A+3;q). \end{aligned} \tag{4.5}$$

It is clear from the definitions that

$$\begin{aligned} T_0(L,A;q) &= T_0(L,-A;q), \\ T_1(L,A;q) &= T_1(L,-A;q). \end{aligned} \tag{4.6}$$

The following limit is also found in [4].

$$\lim_{L\to\infty} U(L,A,q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{4.7}$$

Next, we recall a result of Bailey([9], p. 220):

(4.8)

$$(-z^2q,-z^{-2}q^3,q^4;q^4)_\infty + z(-z^2q^3,-z^{-2}q,q^4;q^4)_\infty = (-z,-z^{-1}q,q;q)_\infty.$$

Finally, we also recall Abel's lemma ([28], page 57):

Lemma 2. *If* $\lim_{n\to\infty} a_n = L$*, then*

$$\lim_{t\to 1_-}(1-t)\sum_{n=0}^{\infty} a_n t^n = L.$$

Proof of Theorem 7. We will show (replacing q by $-q$ in Theorem 7) that

$$\Sigma := \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q;q^2)_n(q^4;q^4)_n}$$
$$= \frac{(q;-q^5)_\infty(-q^4;-q^5)_\infty(-q^5;-q^5)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty} =: \Pi.$$

Define $f(t)$ by

$$f(t,q) := \sum_{n=0}^{\infty} \frac{t^n q^{n^2+2n}(-tq,q^2)_n}{(tq;q^2)_n(-tq^2;q^2)_n(t;q^2)_{n+1}}$$

and define the polynomials $P_n = P_n(q)$ by

$$f(t,q) =: \sum_{n=0}^{\infty} P_n t^n. \tag{4.9}$$

By construction, $\lim_{t\to 1_-}(1-t)f(t,q) = S$ and hence, by Abel's lemma,

$$\lim_{n\to\infty} P_n = \Sigma.$$

We next find a recurrence relation for the P_n.

$$\begin{aligned}
f(t,q) &= \sum_{n=0}^{\infty} \frac{t^n q^{n^2+2n}(-tq,q^2)_n}{(tq;q^2)_n(-tq^2;q^2)_n(t;q^2)_{n+1}}\\
&= \frac{1}{1-t} + \sum_{n=1}^{\infty} \frac{t^n q^{n^2+2n}(-tq,q^2)_n}{(tq;q^2)_n(-tq^2;q^2)_n(t;q^2)_{n+1}}\\
&= \frac{1}{1-t} + \sum_{n=0}^{\infty} \frac{t^{n+1} q^{(n+1)^2+2(n+1)}(-tq,q^2)_{n+1}}{(tq;q^2)_{n+1}(-tq^2;q^2)_{n+1}(t;q^2)_{n+2}}\\
&= \frac{1}{1-t} + \frac{tq^3(1+tq)}{(1-tq)(1+tq^2)(1-t)} \sum_{n=0}^{\infty} \frac{(tq^2)^n q^{n^2+2n}(-tq^3,q^2)_n}{(tq^3;q^2)_n(-tq^4;q^2)_n(tq^2;q^2)_{n+1}}
\end{aligned}$$

$$= \frac{1}{1-t} + \frac{tq^3(1+tq)}{(1-tq)(1+tq^2)(1-t)} f(tq^2, q).$$

Upon clearing denominators, substituting from (4.9) and comparing like powers of t, we have that $P_0 = 1$, $P_1 = 1 + q^3$, $P_2 = 1 + q^3 + 2q^4 + q^8$ and, for $n \geq 3$,

$$P_n = (1+q-q^2+q^{2n+1})P_{n-1} + q(-1+q+q^2+q^{2n-1})P_{n-2} - q^3 P_{n-3}. \tag{4.10}$$

We next find a sequence of polynomials Q_n such that $P_n = Q_n$, for all $n \geq 0$, and $\lim_{n\to\infty} Q_n = \Pi$. In the general case, finding a conjectured form for Q_n involves intelligent guesswork (based on knowledge of what $\lim_{n\to\infty} Q_n$ should be) and some form of "successive approximation" by binomial coefficients, the T_0-, T_1-, U functions, or other similar functions. Once a conjectured form for Q_n has been found, it remains to prove that the conjectured form is the correct one.

We are fortunate in the present case that Sills [23] has found the bosonic representation for the partner identity at (4.2) above (with q replaced by $-q$):

$$Q_n = \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+j} U(n, 5j; q) + q^2 \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+9j} U(n, 5j+2; q).$$

That Q_n above tends to the left side of (4.2) (with q replaced by $-q$) follows from (4.7), the Jacobi triple product identity (2.5) and finally Bailey's identity (4.8) (with q replaced by $-q^5$ and $z = q^2$).

This suggests that we try Q_n of the form

$$Q_n = \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} U(n+a, 5j+b; q) + q^4 (-1)^j q^{10j^2+13j} U(n+a, 5j+c; q),$$

with a, b and c to be determined (It can easily be seen that Q_n tends to the right side of (4.1), for all integers a, b and c, by the same reasoning used immediately above, taking $z = q^4$ in Bailey's identity (4.8)). To determine a, b and c we compute, say, P_5 using (4.10) and then use *Mathematica* to check $P_5 - Q_5$ for a range of values of a, b and c. We find that $a = 1$, $b = 0$ and $c = 3$ appear to work ($P_5 - Q_5 = 0$, for these values). Thus our conjecture is

$$Q_n = \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} U(n+1, 5j; q) + q^4 (-1)^j q^{10j^2+13j} U(n+1, 5j+3; q).$$

One easily checks that $P_n = Q_n$ for $n = 0, 1, 2$. It remains to show that Q_n satisfies the recurrence at (4.10). We need to show that the left side in the first equation below is identically zero.

$$Q_n - (1+q-q^2+q^{2n+1})Q_{n-1} - q(-1+q+q^2+q^{2n-1})Q_{n-2} + q^3 Q_{n-3}$$

$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} \times \Big(U(n+1,5j;q) - (1+q-q^2+q^{2n+1})U(n,5j;q)$$
$$- q(-1+q+q^2+q^{2n-1})U(n-1,5j;q) + q^3 U(n-2,5j;q) \Big)$$
$$+ q^4 (-1)^j q^{10j^2+13j} \times \Big(U(n+1,5j+3;q)$$
$$- (1+q-q^2+q^{2n+1})U(n,5j+3;q)$$
$$- q(-1+q+q^2+q^{2n-1})U(n-1,5j+3;q) + q^3 U(n-2,5j+3;q) \Big)$$
$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} \times$$
$$\Big([U(n+1,5j;q) - (1+q+q^{2n+1})U(n,5j;q) + qU(n-1,5j;q)]$$
$$+ q^2 [U(n,5j;q) - (1+q+q^{2n-1})U(n-1,5j;q) + qU(n-2,5j;q)]$$
$$+ (-1+q)q^{2n} U(n-1,5j;q) \Big)$$
$$+ q^4 (-1)^j q^{10j^2+13j} \times$$
$$\Big([U(n+1,5j+3;q) - (1+q+q^{2n+1})U(n,5j+3;q) + qU(n-1,5j+3;q)]$$
$$+ q^2 [U(n,5j+3;q) - (1+q+q^{2n-1})U(n-1,5j+3;q)$$
$$+ qU(n-2,5j+3;q)] + (-1+q)q^{2n} U(n-1,5j+3;q) \Big)$$
$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} \times$$
$$\Big([q^{2n+2-10j} T_0(n-1,5j-2,q) + q^{2n+2+10j+2} T_0(n-1,5j+3)]$$
$$+ q^2 [q^{2n-10j} T_0(n-2,5j-2,q) + q^{2n+10j+2} T_0(n-2,5j+3)]$$
$$+ (-1+q)q^{2n} U(n-1,5j;q) \Big)$$
$$+ q^4 (-1)^j q^{10j^2+13j} \times$$
$$\Big([q^{2n+2-10j-6} T_0(n-1,5j+1,q) + q^{2n+2+10j+8} T_0(n-1,5j+6)]$$
$$+ q^2 [q^{2n-10j-6} T_0(n-2,5j+1,q) + q^{2n+10j+8} T_0(n-2,5j+6)]$$
$$+ (-1+q)q^{2n} U(n-1,5j+3;q) \Big).$$

The last identity follows from (4.5). Thus, after cancelling a factor of q^{2n}, proving that Q_n satisfies the stated recurrence relation comes down to showing that

$$
\begin{aligned}
&(1-q)\sum_{j=-\infty}^{\infty}(-1)^j q^{10j^2+3j}U(n-1,5j;q)+q^4(-1)^j q^{10j^2+13j}U(n-1,5j+3;q)\\
&=\sum_{j=-\infty}^{\infty}(-1)^j q^{10j^2+3j}\bigg(q^{2-10j}T_0(n-1,5j-2,q)+q^{10j+4}T_0(n-1,5j+3)\\
&\qquad\qquad+q^{2-10j}T_0(n-2,5j-2,q)+q^{10j+4}T_0(n-2,5j+3)\bigg)\\
&\quad+q^4(-1)^j q^{10j^2+13j}\bigg(q^{-10j-4}T_0(n-1,5j+1,q)+q^{10j+10}T_0(n-1,5j+6)\\
&\qquad\qquad+q^{-10j-4}T_0(n-2,5j+1,q)+q^{10j+10}T_0(n-2,5j+6)\bigg)\\
&=\sum_{j=-\infty}^{\infty}(-1)^j\bigg(q^{10j^2-7j+2}T_0(n-1,5j-2,q)+q^{10j^2+13j+4}T_0(n-1,5j+3)\\
&\qquad\qquad+q^{10j^2-7j+2}T_0(n-2,5j-2,q)+q^{10j^2+13j+4}T_0(n-2,5j+3)\bigg)\\
&\quad+(-1)^j\bigg(q^{10j^2+3j}T_0(n-1,5j+1,q)+q^{10j^2+23j+14}T_0(n-1,5j+6)\\
&\qquad\qquad+q^{10j^2+3j}T_0(n-2,5j+1,q)+q^{10j^2+23j+14}T_0(n-2,5j+6)\bigg)\\
&=\sum_{j=-\infty}^{\infty}(-1)^j\bigg(-q^{10j^2+13j+5}T_0(n-1,5j+3,q)+q^{10j^2+13j+4}T_0(n-1,5j+3)\\
&\qquad\qquad-q^{10j^2+13j+5}T_0(n-2,5j+3,q)+q^{10j^2+13j+4}T_0(n-2,5j+3)\bigg)\\
&\quad+(-1)^j\bigg(q^{10j^2+3j}T_0(n-1,5j+1,q)-q^{10j^2+3j+1}T_0(n-1,5j+1)\\
&\qquad\qquad+q^{10j^2+3j}T_0(n-2,5j+1,q)-q^{10j^2+3j+1}T_0(n-2,5j+1)\bigg).
\end{aligned}
$$

The last equality follows upon replacing j by $j+1$ in the first and third sums, and replacing j by $j-1$ in the sixth and eighth sums. Hence (after cancelling a factor of $1-q$) what is needed is to show that

$$
\sum_{j=-\infty}^{\infty}(-1)^j q^{10j^2+3j}U(n-1,5j;q)+(-1)^j q^{10j^2+13j+4}U(n-1,5j+3;q)
$$

$$= \sum_{j=-\infty}^{\infty} (-1)^j \left(q^{10j^2+13j+4} T_0(n-1, 5j+3) + q^{10j^2+13j+4} T_0(n-2, 5j+3) \right)$$
$$+ (-1)^j \left(q^{10j^2+3j} T_0(n-1, 5j+1, q) + q^{10j^2+3j} T_0(n-2, 5j+1, q) \right).$$

Alternatively, after substituting for $U(n-1, 5j; q)$ and $U(n-1, 5j+3; q)$ from (4.3), we see that what remains is to show that

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} (T_0(n-1, 5j, q) - T_0(n-2, 5j+1))$$
$$+ (-1)^j q^{10j^2+13j+4} (T_0(n-1, 5j+4, q) - T_0(n-2, 5j+3)) = 0,$$

or, upon replacing n with $n+1$, that

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} (T_0(n, 5j, q) - T_0(n-1, 5j+1))$$
$$+ (-1)^j q^{10j^2+13j+4} (T_0(n, 5j+4, q) - T_0(n-1, 5j+3)) = 0.$$

We now show that this does indeed hold.

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+3j} (T_0(n, 5j, q) - T_0(n-1, 5j+1))$$
$$+ (-1)^j q^{10j^2+13j+4} (T_0(n, 5j+4, q) - T_0(n-1, 5j+3))$$
$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2-3j} (T_0(n, 5j, q) - T_0(n-1, 5j-1))$$
$$+ (-1)^j q^{10j^2+13j+4} (T_0(n, 5j+4, q) - T_0(n-1, 5j+3))$$
$$= \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2-3j} (q^{n+5j} T_1(n-1, 5j, q) + q^{2n+10j} T_0(n-1, 5j+1)) +$$
$$(-1)^j q^{10j^2+13j+4} (q^{n+5j+4} T_1(n-1, 5j+4, q) + q^{2n+10j+8} T_0(n-1, 5j+5))$$
$$= q^n \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+2j} (T_1(n-1, 5j, q) + q^{n+5j} T_0(n-1, 5j+1))$$
$$+ (-1)^j q^{10j^2+18j+8} (T_1(n-1, 5j+4, q) + q^{n+5j+4} T_0(n-1, 5j+5))$$
$$= q^n \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+2j} (T_1(n-1, 5j, q) + q^{n+5j} T_0(n-1, 5j+1))$$
$$- (-1)^j q^{10j^2+2j} (T_1(n-1, 5j+1, q) + q^{n-5j-1} T_0(n-1, 5j))$$

$$= q^n \sum_{j=-\infty}^{\infty} (-1)^j q^{10j^2+2j} \bigg(T_1(n-1,5j,q) + q^{n+5j} T_0(n-1,5j+1) \\ - T_1(n-1,5j+1,q) - q^{n-5j-1} T_0(n-1,5j) \bigg)$$
$$= 0.$$

The first equality follows from (4.6) and replacing j by $-j$ in the first two sums; the second equality follows from the first relation at (4.4); the fourth equality follows upon applying (4.6) to the third and fourth sums and then replacing j by $-j-1$; the last equality follows from the second relation at (4.4). □

Remarks: 1) We mention in passing that Andrew Sills has created a Maple package, "`RRtools`", which automates much of the process involved in searching for the bosonic representations Q_n (see [24]).

2) Once the conjectured form for Q_n had been found, Zeilberger's algorithm, as implemented in the *Mathematica* packages developed by Axel Riese (see [17] and [19]), could have been used to find a recurrence for Q_n (hopefully the minimal recurrence, thereby showing that Q_n satisfies the recurrence at (4.10)). However, this method, although extremely effective, lacks transparency for those not familiar with it and we preferred to prove our result as we did.

5. Miscellaneous Methods

The final identity which the present investigations uncovered is the following:

Theorem 8. *Let $|q| < 1$. Then*

$$\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q;q)_n}{(q;q^2)_{n+1}(q;q)_{n+1}} = \frac{(q^{10};q^{10})_\infty}{(q;q)_\infty (q;q^2)_\infty (-q^3,-q^4,-q^6,-q^7;q^{10})_\infty}. \tag{5.1}$$

We were unable to apply the methods of the previous sections to this identity and had to rely for its proof on two fairly recent results.

The first of these is an Identity of Blecksmith, Brillhart and Gerst [10] (a proof is also given in [12]):

$$\sum_{n=-\infty}^{\infty} q^{n^2} - \sum_{n=-\infty}^{\infty} q^{5n^2} = 2q \frac{(q^4,q^6,q^{10},q^{14},q^{16},q^{20};q^{20})_\infty}{(q^3,q^7,q^8,q^{12},q^{13},q^{17};q^{20})_\infty}. \tag{5.2}$$

If we replace q by $-q$ and apply the Jacobi triple product identity to the left side, (5.2) may be re-written as

$$(q^5,q^5,q^{10};q^{10})_\infty - (q,q,q^2;q^2)_\infty = 2q \frac{(q^4,q^6,q^{10};q^{10})_\infty}{(-q^3,-q^7;q^{10})_\infty (q^8,q^{12};q^{20})_\infty}. \tag{5.3}$$

The second is the following identity, proved by Sills [25]:

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q)_n(q;q^2)_n} = \frac{(q^5,q^5,q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty}. \tag{5.4}$$

Quite remarkably, (5.4) does not appear on Slater's list and its first appearance in print appears to be in [25].

Proof of Theorem 8.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{(n^2+3n)/2}(-q;q)_n}{(q;q^2)_{n+1}(q;q)_{n+1}} &= \frac{1}{2q}\sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}(-1;q)_{n+1}}{(q;q^2)_{n+1}(q;q)_{n+1}} \\
&= \frac{1}{2q}\left(\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-1;q)_n}{(q;q^2)_n(q;q)_n} - 1\right) \\
&= \frac{1}{2q}\left(\frac{(q^5,q^5,q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty} - 1\right) \text{ (by (5.4))} \\
&= \frac{1}{2q(q;q)_\infty(q;q^2)_\infty}\big((q^5,q^5,q^{10};q^{10})_\infty - (q;q^2)_\infty(q;q^2)_\infty(q^2;q^2)_\infty\big) \\
&= \frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty(-q^3,-q^7;q^{10})_\infty(q^8,q^{12};q^{20})_\infty} \text{ (by (5.3))} \\
&= \frac{(q^{10};q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty(-q^3,-q^4,-q^6,-q^7;q^{10})_\infty}.
\end{aligned}$$

□

6. Concluding Remarks

The proof of Theorem 7 is quite long, even if it essentially involves just several re-indexing of series and using known identities. For quite a long time we were convinced that there must exist another general transformation of the type found in Theorem 1 or in Watson's theorem (see (3.1)), a transformation which would give the result in Theorem 7 as a special case for particular values of its parameters.

One reason we thought this transformation had to exist was the appearance of the $(-q;q^5)_\infty(-q^4;q^5)_\infty(q^5;q^5)_\infty$ term on the product side, which can be represented as an infinite series via the Jacobi triple product. This in turn brought to mind Watson's proof of the Rogers-Ramanujan identities, where he showed that these followed as special cases of (3.1).

However, we could not find such a transformation, but possibly our search was incomplete. Does the identity in Theorem 7 follow as a special case of some known transformation, perhaps some known transformation between basic hypergeometric series? Is this identity a special case of some as yet undiscovered general transformation?

In a subsequent paper we will give finite forms of the identities in sections 2 and 3. We will also examine the dual identities that can be derived from these finite forms.

References

[1] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976.

[2] G. E. Andrews, *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra.* CBMS Regional Conference Series in Mathematics, **66**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. xii+130 pp.

[3] Andrews, George E. *Euler's "exemplum memorabile inductionis fallacis" and q-trinomial coefficients.* J. Amer. Math. Soc. **3** (1990), no. 3, 653–669.

[4] Andrews, George E.; Baxter, R. J. *Lattice gas generalization of the hard hexagon model. III. q-trinomial coefficients.* J. Statist. Phys. **47** (1987), no. 3-4, 297–330.

[5] George E. Andrews, Bruce C. Berndt. *Ramanujan's lost notebook. Part I.* Springer, New York, 2005. xiv+437 pp.

[6] W.N. Bailey, *A note on certain q-identities*, Q. J. Math. **12** (1941) 173-175.

[7] Bailey, W. N. *Some Identities in Combinatory Analysis.* Proc. London Math. Soc. **49** (1947) 421–435.

[8] Bailey, W. N, *Identities of the Rogers-Ramanujan type.* Proc. London Math. Soc., **50** (1949) 421–435.

[9] Bailey, W. N. *On the simplification of some identities of the Rogers-Ramanujan type.* Proc. London Math. Soc. (3) **1**, (1951). 217–221.

[10] Blecksmith, Richard; Brillhart, John; Gerst, Irving *Some infinite product identities.* Math. Comp. **51** (1988), no. 183, 301–314.

[11] A. L. Cauchy, *Mémoire sur les fonctions dont plusieurs valeurs* , C. R. Acad. Sci. Paris **17** (1843), 523; reprinted in Oeuvres de Cauchy, Ser. 1 **8**, Gauthier-Villars, Paris (1893), 42-50.

[12] Cooper, Shaun; Hirschhorn, Michael *On some infinite product identities.* Rocky Mountain J. Math. **31** (2001), no. 1, 131–139.

[13] Göllnitz, H. *Partitionen mit Differenzenbedingungen.* J. reine angew. Math. **225** (1967) 154–190.

[14] Basil Gordon,*Some continued fractions of the Rogers-Ramanujan type.* Duke Math. J. **32** 1965 741–748.

[15] Hardy, G. H. (Godfrey Harold). *Lectures by Godfrey H. Hardy on the mathematical work of Ramanujan ; fall term 1936 / Notes by Marshall Hall.* The Institute for advanced study. Ann Arbor, Mich., Edwards bros., inc., 1937.

[16] Jackson, F. H. *Examples of a Generalization of Euler's Transformation for Power Series.* Messenger Math. **57** (1928) 169–187.

[17] Paule, Peter; Riese, Axel *A Mathematica q-analogue of Zeilberger's algorithm based on an algebraically motivated approach to q-hypergeometric telescoping.* Special functions, q-series and related topics (Toronto, ON, 1995), 179–210, Fields Inst. Commun., **14**, Amer. Math. Soc., Providence, RI, 1997.

[18] S. Ramanujan, *The lost notebook and other unpublished papers.* With an introduction by George E. Andrews. Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988. xxviii+419 pp. 45

[19] Riese, Axel *qMultiSum—a package for proving q-hypergeometric multiple summation identities.* J. Symbolic Comput. **35** (2003), no. 3, 349–376.

[20] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math.Soc. **25** (1894),pp. 318-343.

[21] L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math.Soc. **16** (1917),pp. 315-336.

[22] Issai Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüchen*, in *Gesammelte Abhandlungen. Band II*, Springer-Verlag, Berlin-New York, 1973, 117-136.
(Originally in *Sitzungsberichte der Preussischen Akadamie der Wissenschaften*, 1917, Physikalisch-Mathematische Klasse, 302-321)

[23] Sills, Andrew V. *Finite Rogers-Ramanujan type identities.* Electronic J. Combin. **10(1)** (2003) #R13, 1-122.

[24] Sills, Andrew V. `RRtools`—*a Maple package for aiding the discovery and proof of finite Rogers-Ramanujan type identities.* J. Symbolic Comput. **37** (2004), no. 4, 415–448.

[25] Sills, Andrew V. *On identities of the Rogers-Ramanujan type.* to appear in the Ramanujan Journal.

[26] Slater, L. J. *Further identities of the Rogers-Ramanujan type*, Proc. London Math.Soc. **54** (1952) 147–167.

[27] Watson, G. N. *A New Proof of the Rogers-Ramanujan Identities.* J. London Math. Soc. **4**, 4-9, 1929.

[28] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis,* Cambridge University Press, 4th edition, 1952.

Northern Illinois University, Mathematical Sciences, DeKalb, IL 60115-2888

E-mail address: `bowman@math.niu.edu`

Mathematics Department, Anderson Hall, West Chester University, West Chester, PA 19383

E-mail address: `jmclaughl@wcupa.edu`